

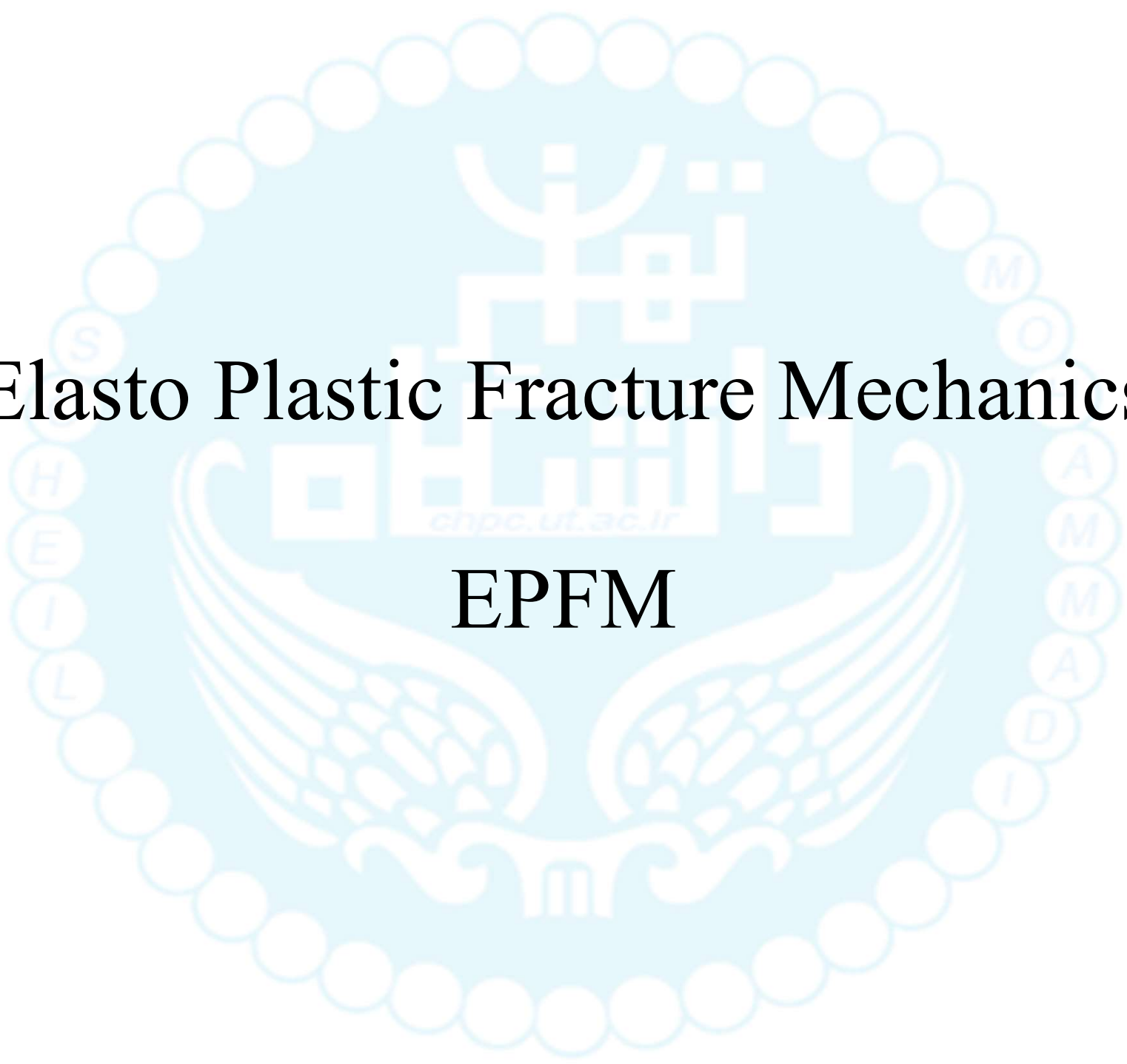


Fracture Mechanics

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Elasto Plastic Fracture Mechanics

EPFM

Elstoplastic Fracture Mechanics, EPFM

Under the assumptions of linear elastic fracture mechanics, the stress at the crack tip is theoretically infinite. Consequently, it may usually lead to conservative and expensive solutions as it does not account for plastification at the crack tip.

From a physical point of view, however, no material can withstand infinite stress, and a small plastic/fractured zone will be formed around the crack tip. As a result, an extension to ductile fracture or elastoplastic fracture mechanics (EPFM) is required.

Elstoplastic Fracture Mechanics, EPFM

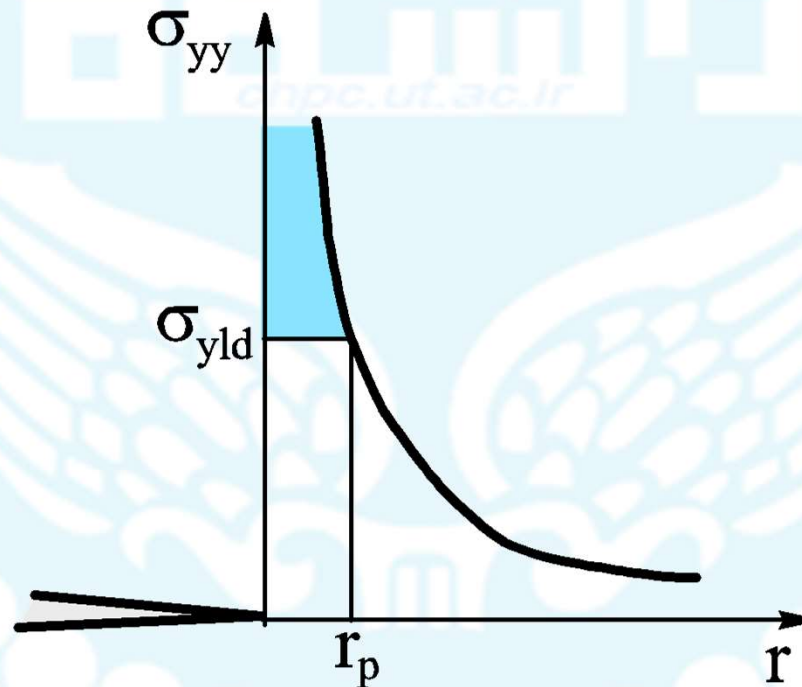
If this zone is small compared to the crack size, then the linear elastic assumptions are correct. Otherwise LEFM is not applicable (thus it would be incorrect to use K) and a nonlinear model in terms of G must be used. Usually, such a “damaged” zone is referred to as a plastic zone for metals, and a fracture process zone for cementitious materials and ceramics.

There are two important issues associated with nonlinear fracture:

1. What is the size of the plastic or process zone?
2. What are the criteria for crack growth?

First-order uniaxial stress criterion

Here, only the uniaxial stress state normal to the crack axis is considered. The simplest estimate of the size of the plastic zone is obtained by equating the uniaxial stress σ_{yy} to the yield stress σ_{yld} .



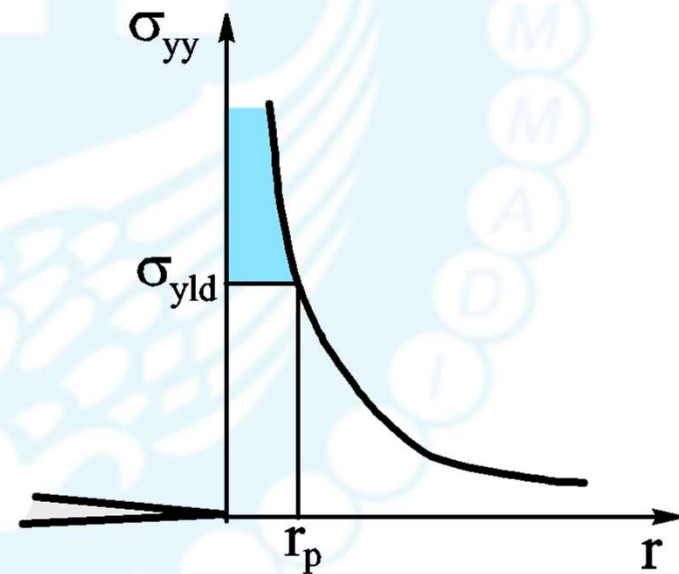
For plane stress problems,

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)$$

the size of the plastic zone, $r=r_p$, can be obtained at $\theta=0$ from

$$\sigma_{yy} \Big|_{\substack{r=r_p \\ \theta=0}} = \frac{K_I}{\sqrt{2\pi r_p}} = \sigma_{yld}$$

$$r_p = \frac{1}{2\pi} \frac{K_I^2}{\sigma_{yld}^2}$$



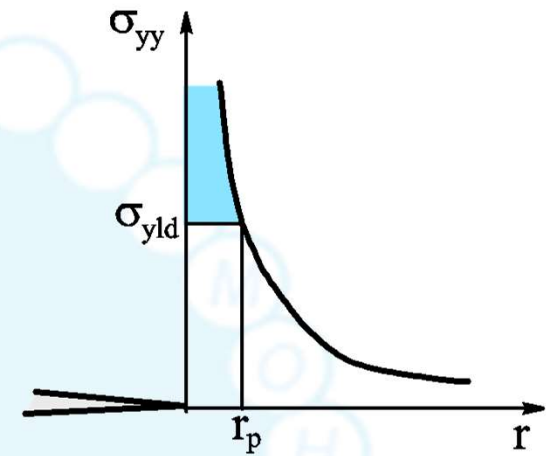
or in terms of the stress intensity factor for a central crack in an infinite plate $K_I = \sigma_0 \sqrt{\pi a}$

$$r_p = \frac{a}{2} \left(\frac{\sigma_0}{\sigma_{yld}} \right)^2$$

Similarly, the size of the plastic zone for mode II for both plane stress and plane strain problems can be obtained from:

$$r_p = \frac{3}{2\pi} \frac{K_{II}^2}{\sigma_{yld}^2}$$

The main problem with this approximation is that it simply ignores all stresses exceeding σ_{yld} . As a result, equilibrium is no longer satisfied.



Note that, for $\sigma_0 = \sigma_{yld}$

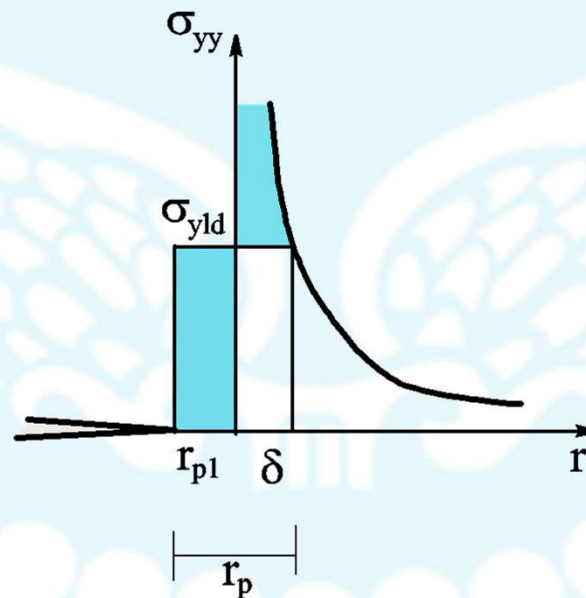
- The whole plate is in fact in a yielding tensile stress state, while the formulation predicts the size of crack tip plastic zone to be a limited value:

$$\sigma_0 = \sigma_{yld} \rightarrow r_p = \frac{a}{2}$$

- This relatively large value of r_p still violates the assumption of small plastic zone of LEFM.

Second-order uniaxial stress criterion

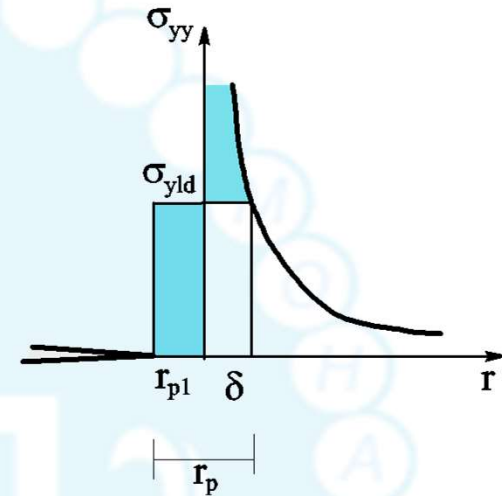
In an alternative approach to avoid violation of equilibrium equations, Irwin (1960) developed a second-order approximation for the plastic zone based on the stress redistribution occurring at the crack tip. In this model, the area under the stress curve which was eliminated in the first order approach, is redistributed to satisfy equilibrium requirements.



$$\int_0^{\delta} \frac{K_I}{\sqrt{2\pi r}} dr = r_p \sigma_{yld} = (\delta + r_{p1}) \sigma_{yld}$$

The final solution is then obtained as:

$$\delta = r_{p1} = \frac{1}{2\pi} \frac{K_I^2}{\sigma_{yld}^2}$$



or in terms of K_I for a central crack in an infinite plate

$$\delta = r_{p1} = \frac{a}{2} \left(\frac{\sigma_0}{\sigma_{yld}} \right)^2$$

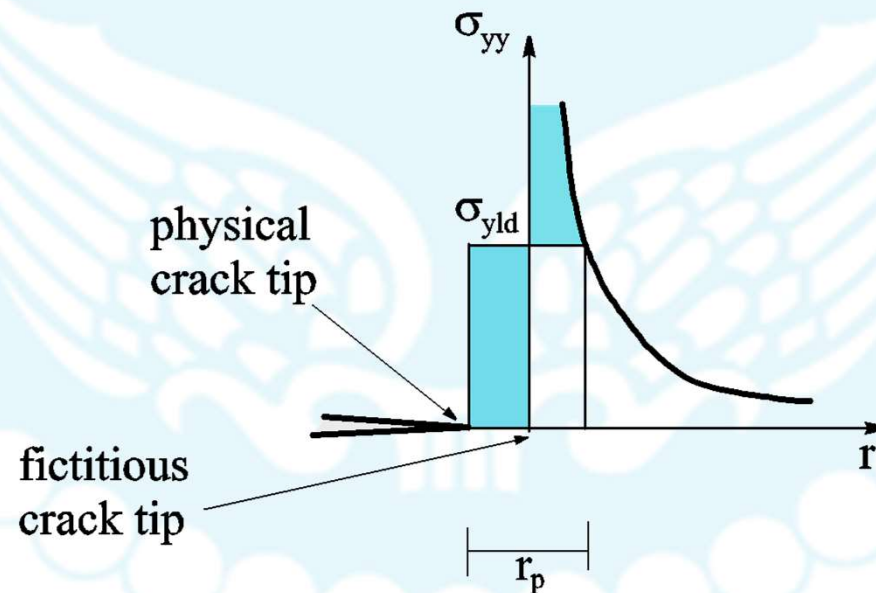
which means that

$$r_p = 2r_{p1} = 2\delta$$

implying that the second-order size is twice the first-order size. Thus, the first-order solution r_{p1} may still be used with an effective crack length of $\bar{a} = a + r_{p1}$ extending to the centre of the plastic zone. Thus,

$$\bar{a} = a + r_{p1}$$

$$K_I = \sigma_0 \sqrt{\pi \bar{a}} = \sigma_0 \sqrt{\pi (a + r_{p1})}$$



Note that, for $\sigma_0 = \sigma_{\text{yld}}$

- The whole plate is in fact in a yielding tensile stress state, while the formulation predicts the size of crack tip plastic zone to be a limited value:

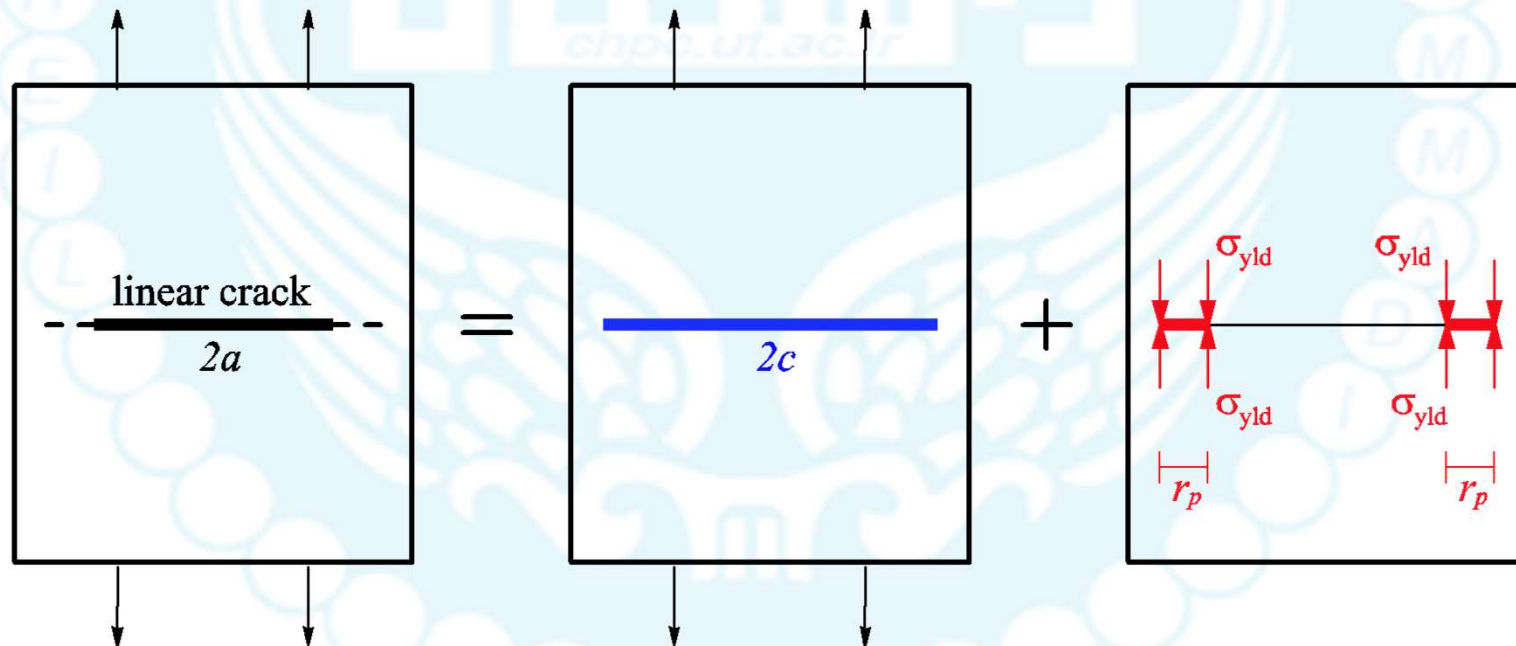
$$\sigma_0 = \sigma_{\text{yld}} \rightarrow r_p = a$$

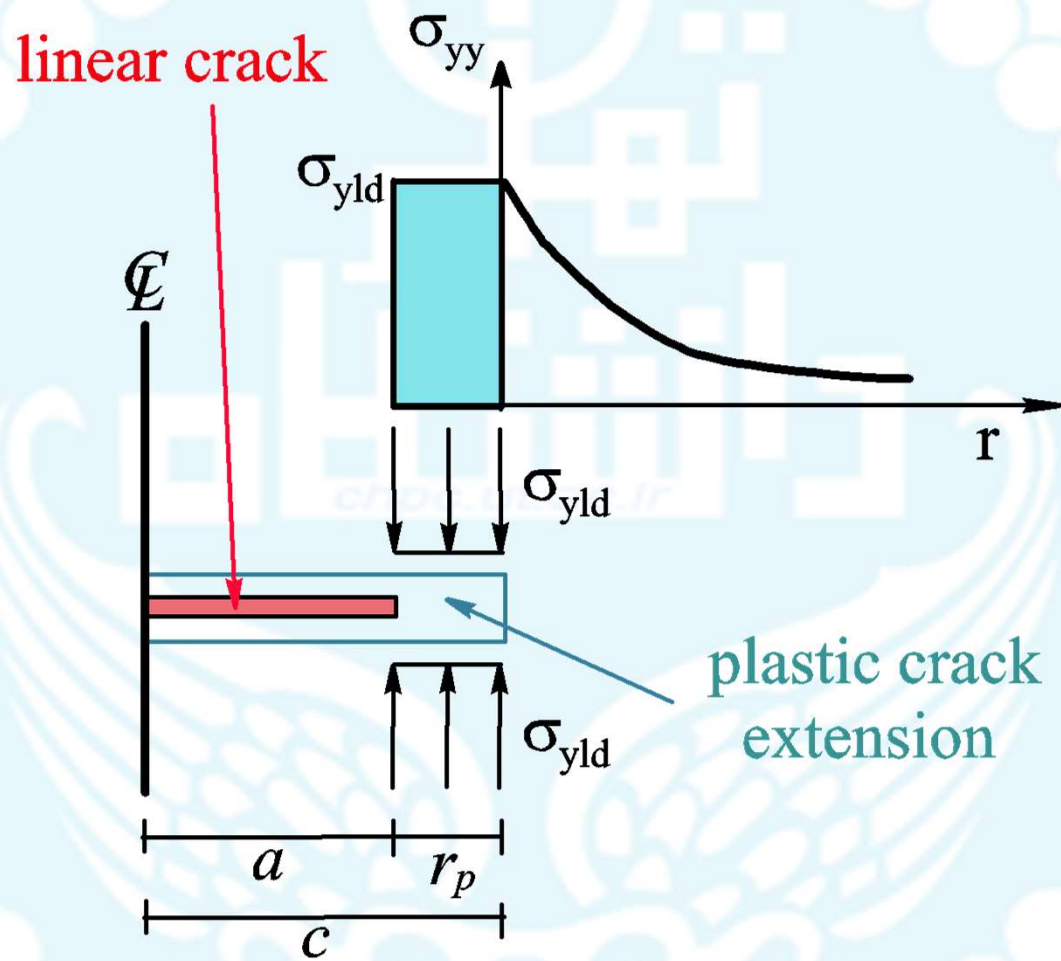
- This relatively large value of r_p still violates the assumption of small plastic zone of LEFM.

Dugdale uniaxial model

Another alternative to satisfy equilibrium equations was proposed by Dugdale (1960) based on the assumption of replacing the actual physical crack length $2a$ by a total effective crack length $2c$,

$$c = a + r_p$$

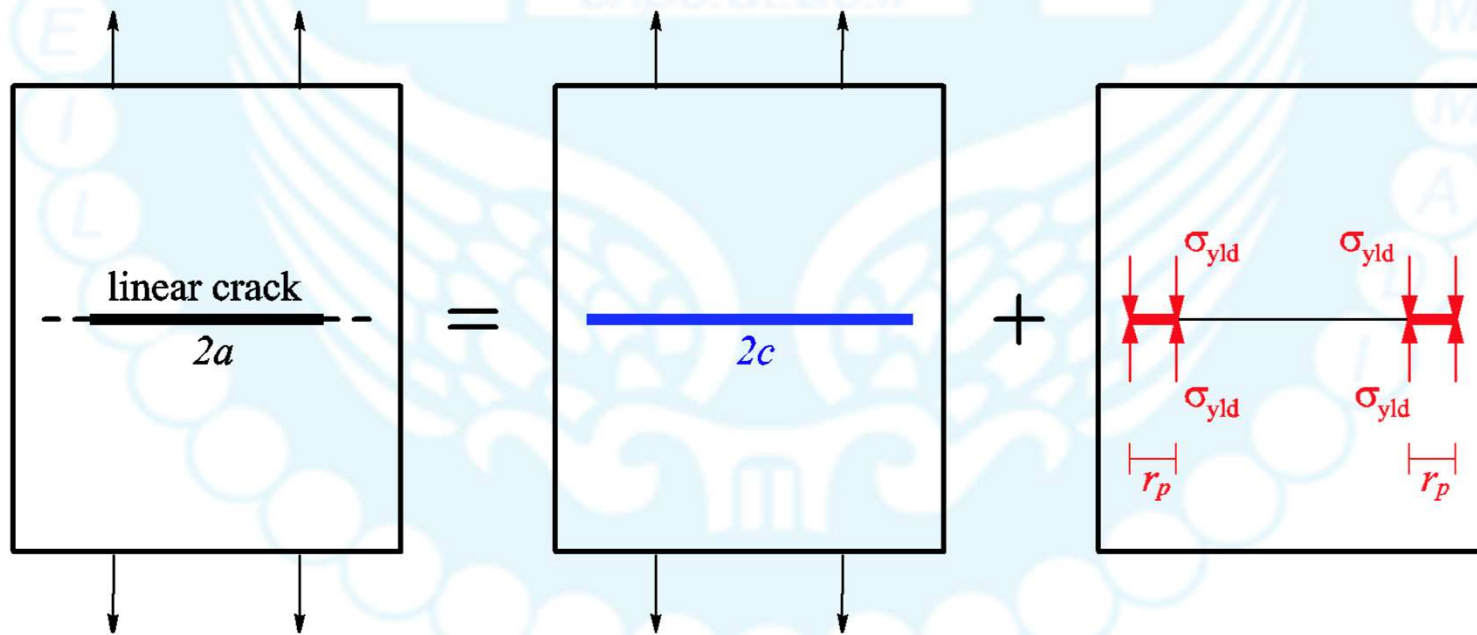


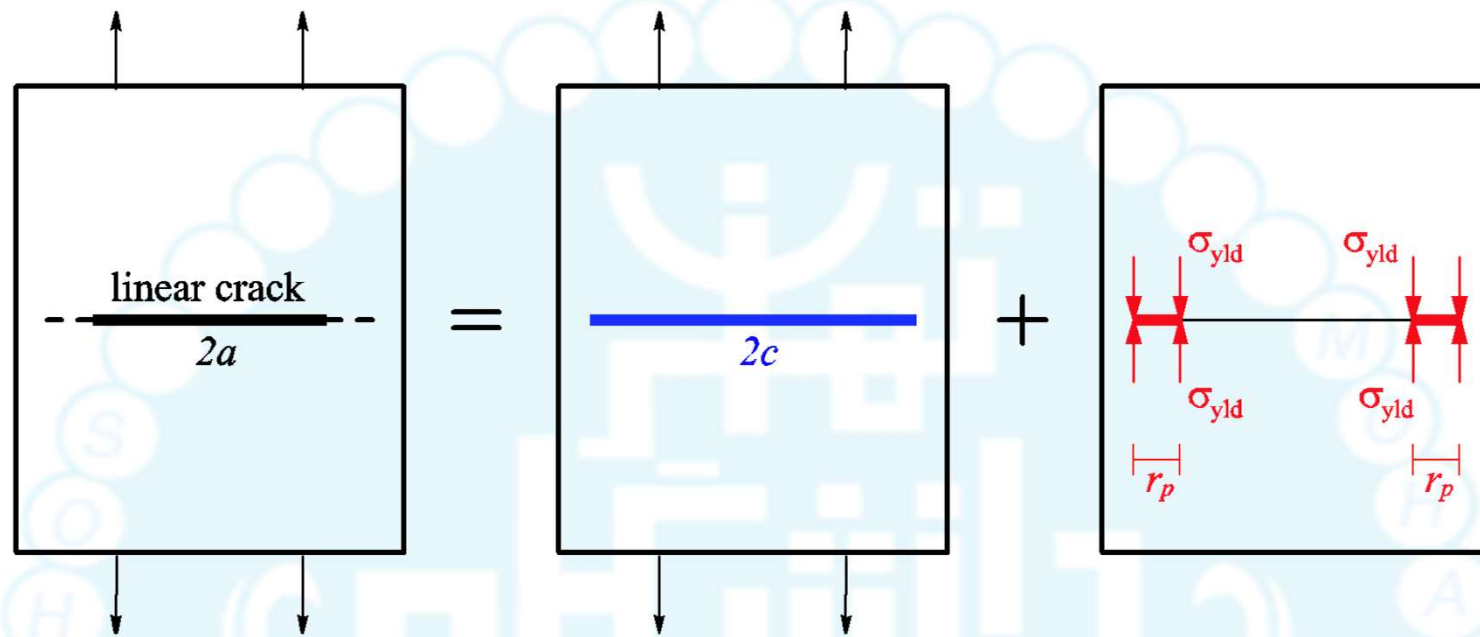


Dugdale assumed that a closing constant stress σ_{yld} is applied over the length r_p , causing a negative K_{yld} . As a result, the combined model requires the overall stress intensity factor to vanish:

$$K_{combined} = K + K_{\sigma_{yld}} = 0$$

so determining the length c or r_p .





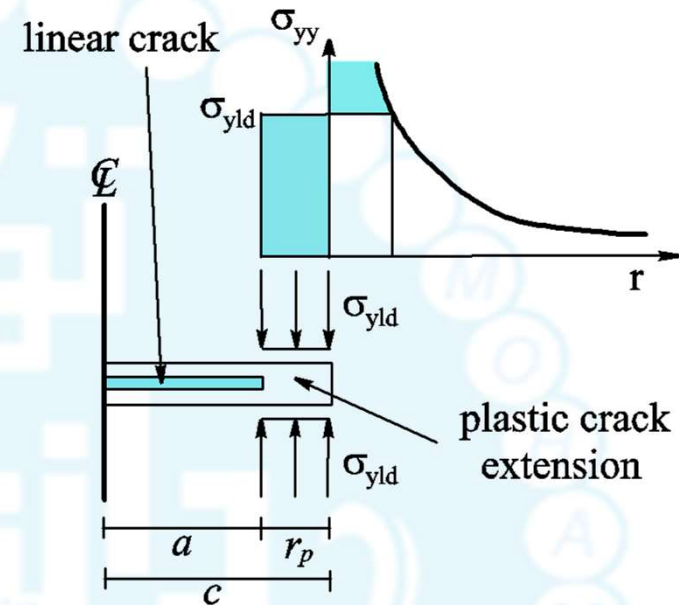
$$K_{\text{combined}} = K + K_{\sigma_{yld}} = 0$$

$$K = \sigma_0 \sqrt{\pi c}$$

$$K_{\sigma_{yld}} = -2\sigma_{yld} \sqrt{\frac{c}{\pi}} \cos^{-1} \frac{a}{c}$$

Results in:

$$\frac{a}{c} = \cos \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_{yld}} \right)$$



Then, applying the Taylor's expansion and neglecting higher order terms results in (for plane stress):

$$r_p = \frac{\pi^2}{8} \left(\frac{\sigma_0}{\sigma_{yld}} \right)^2 a = \frac{\pi}{8} \left(\frac{K_I}{\sigma_{yld}} \right)^2$$

and for plane strains σ_{yld} is replaced by $\sqrt{3}\sigma_{yld}$.

It is important to note that for equation

$$\frac{a}{c} = \cos \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_{yld}} \right)$$

If the loading σ_0 equals the yield stress σ_{yld} :

$$\sigma_0 \rightarrow \sigma_{yld} : \frac{a}{c} \rightarrow 0, \quad c, r_p \rightarrow \infty$$

the plastic radius becomes infinite and the plasticity extends to whole plate. While this is in agreement with the physical fully plastic state of the plate, it clearly contradicts the assumption of small plastic zone of EPFM.

On the contrary, for the simplified equation based on neglecting higher order terms of the Taylor's series :

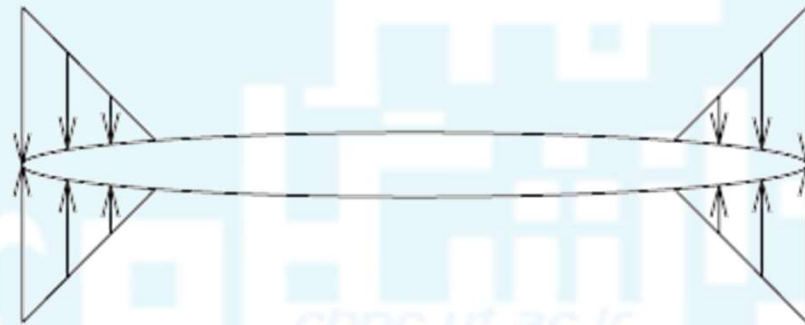
$$r_p = \frac{\pi^2}{8} \left(\frac{\sigma_0}{\sigma_{yld}} \right)^2 \quad a = \frac{\pi}{8} \left(\frac{K_I}{\sigma_{yld}} \right)^2$$

If the loading σ_0 approaches the yield stress σ_{yld} :

$$\sigma_0 \rightarrow \sigma_{yld} : \quad r_p \rightarrow \frac{\pi^2}{8} a$$

This is closer to the results of second-order theory.

An extension to the model was proposed by Barenblatt, 1962.



Dugdale and Barenblatt models are the bases for a number of cohesive crack models, to be discussed in future.

First-order multiaxial yield criterion

The idea of a plastic zone can further be extended to the first-order multiaxial conditions to include points other than $\theta = 0$. The idea is to assume that yielding would occur when the effective stress σ_{eff} computed from any specified yield criterion reaches σ_{yld} .

In order to provide sample explicit solutions, the von Mises yield criterion in terms of principal stresses σ_1 , σ_2 and σ_3 is considered:

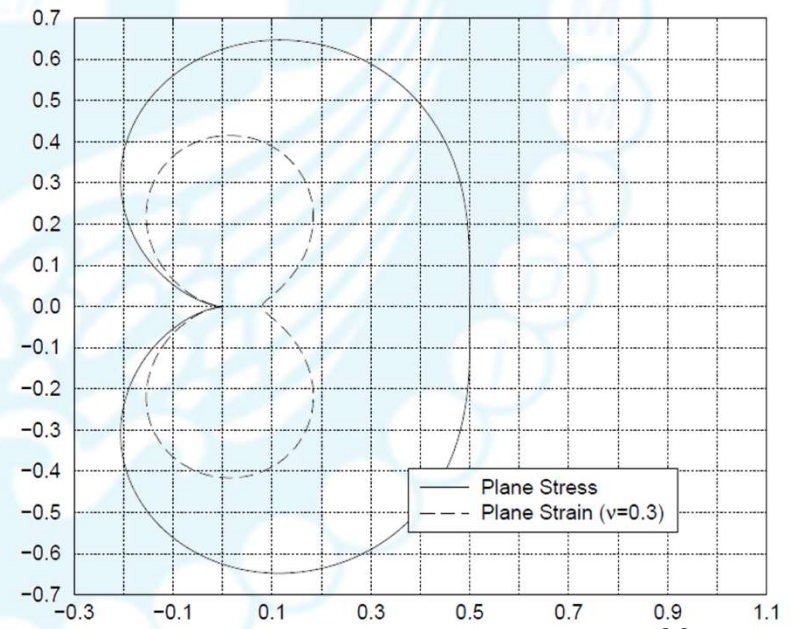
$$\sigma_{\text{eff}} = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{\frac{1}{2}}$$

The size of the plastic zone $r_p(\theta)$ can be evaluated for plane strain problems as:

$$r_p(\theta) = \frac{1}{4\pi} \frac{K_I}{\sigma_{yld}^2} \left[\frac{3}{2} \sin^2 \theta + (1 - 2\nu)^2 (1 + \cos \theta) \right]$$

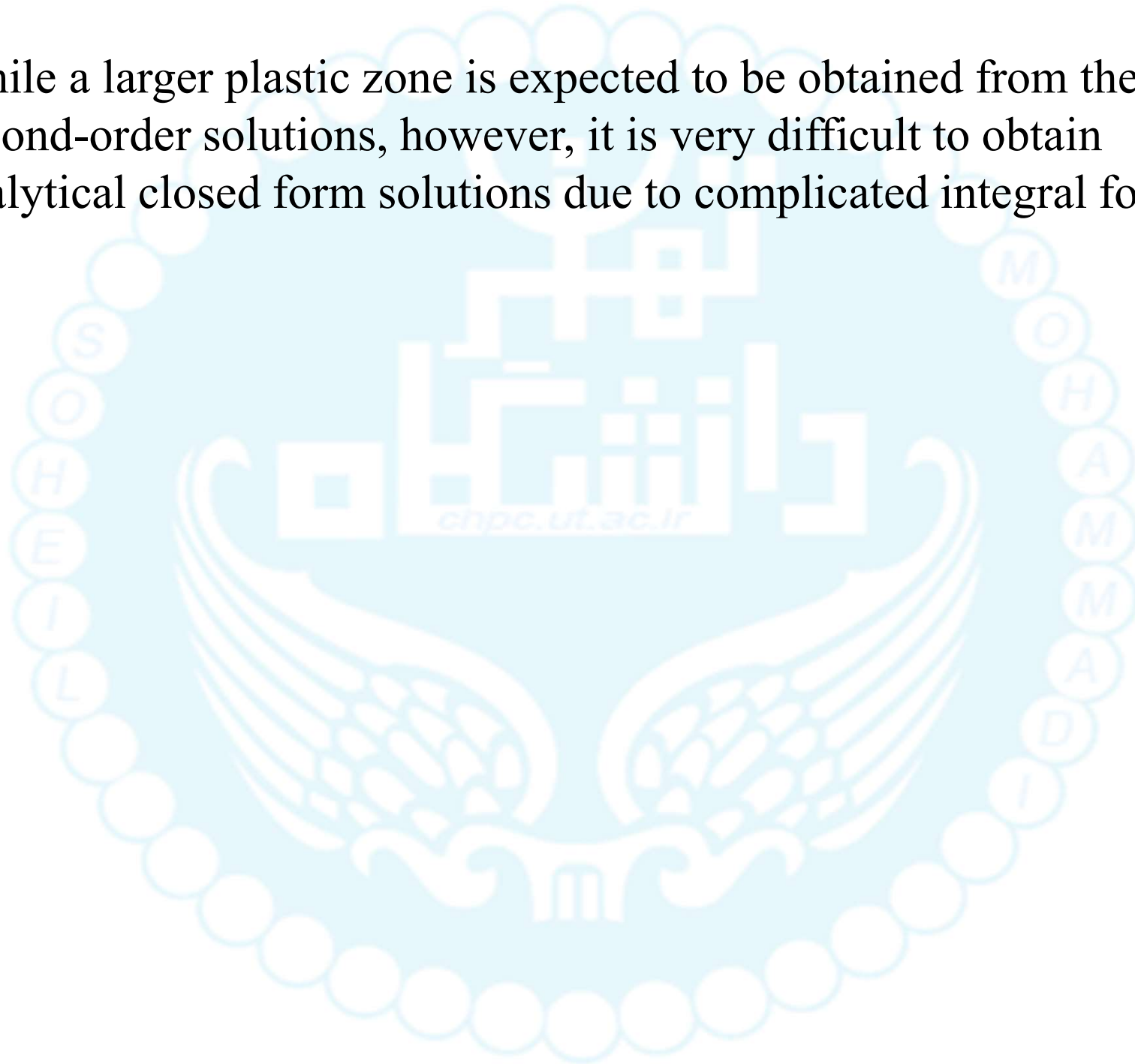
and a much larger size for plane stress conditions (WHY?):

$$r_p(\theta) = \frac{1}{4\pi} \frac{K_I}{\sigma_{yld}^2} \left[1 + \frac{3}{2} \sin^2 \theta + \cos \theta \right]$$



Normalized Mode I Plastic Zone (von Mises)

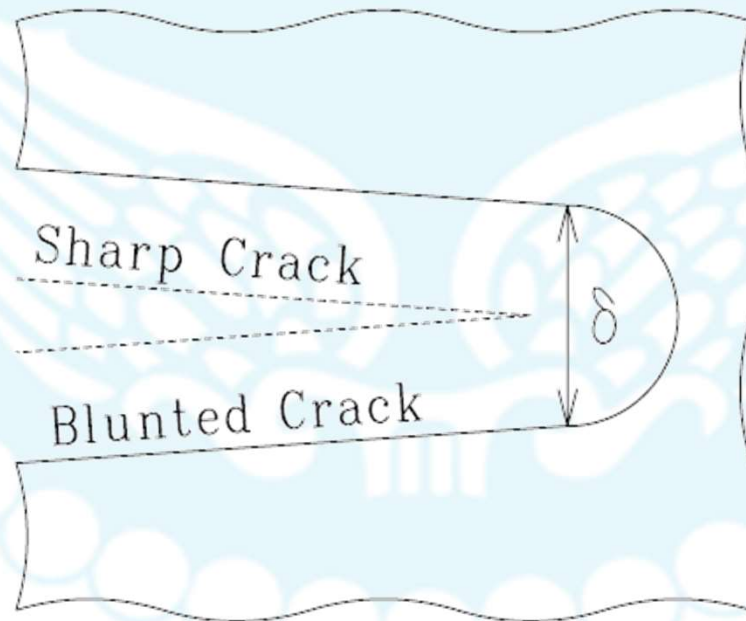
While a larger plastic zone is expected to be obtained from the second-order solutions, however, it is very difficult to obtain analytical closed form solutions due to complicated integral forms



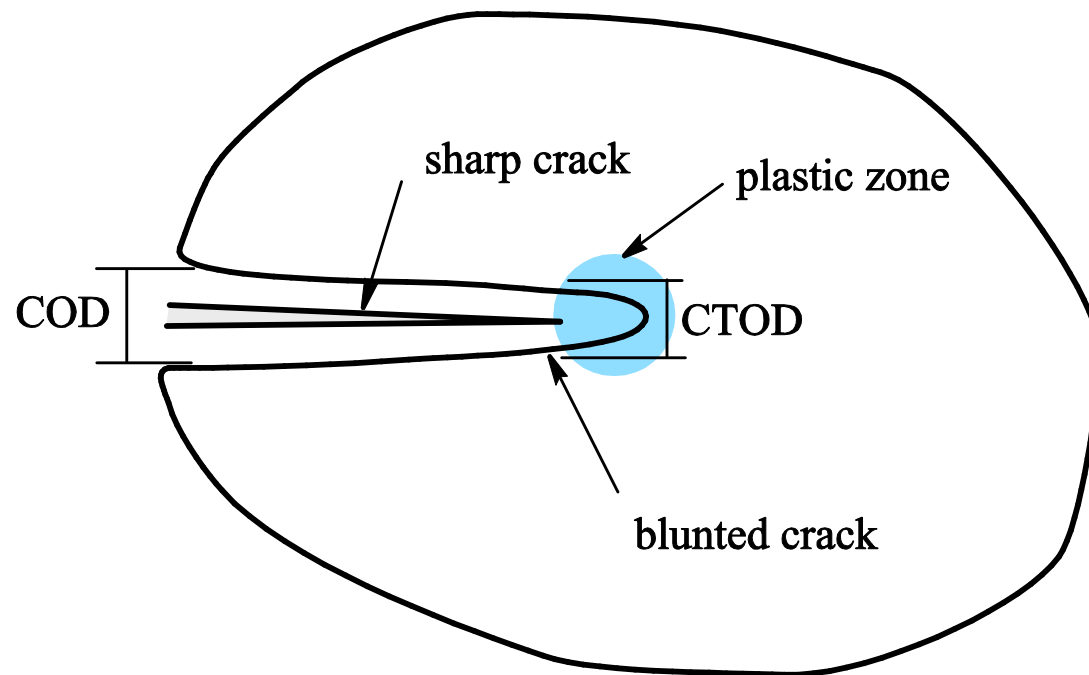
Crack tip opening displacements (CTOD)

In a totally different approach, a local criterion based on the crack tip opening displacement (CTOD) has been proposed to account for elastoplastic behaviour around the crack tip (Cottrell, 1961 and Wells, 1963).

COD and CTOD criteria are frequently used in British codes of practice.



In LEFM and brittle fracture, sharp cracks are considered and the CTOD is always zero. In contrast, CTOD is not negligible in ductile fracture and elastoplastic fracture mechanics due to relatively large deformation and blunting of the crack.



Similar to the method of determining the size of the plastic zone, two first- and second order CTOD approaches are available. ²⁵

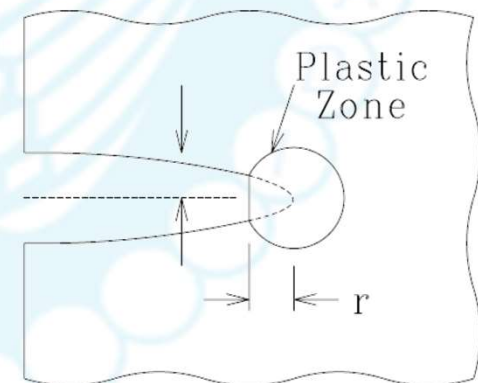
First-order CTOD

The first-order solution for CTOD is based on the Irwin's solution for the mode I vertical displacement of a point next to the crack tip.

$$u_y = \frac{K_I}{2\mu} \left[\frac{r}{2\pi} \right]^{\frac{1}{2}} \sin \frac{\theta}{2} \left[\kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right]$$

Substituting $\theta = \pm\pi$ results in evaluation of crack tip opening (COD) at a distance r .

$$\text{COD} = 2u_y = \frac{\kappa + 1}{\mu} K_I \sqrt{\frac{r}{2\pi}}$$



Then, for $r=r_p$, where r_p is the Irwin plastic zone, the final crack tip opening (CTOD) is estimated as:

$$r_p = \frac{1}{2\pi} \frac{K_I^2}{\sigma_{yld}^2}$$

$$CTOD = \frac{4}{\pi} \frac{K_I^2}{E \sigma_{yld}}$$

Recalling the definition of the energy release rate, the following approximation can be assumed:

$$CTOD = \frac{4}{\pi} \frac{K_I^2}{E \sigma_{yld}} \xrightarrow{G = \frac{K_I^2}{E'}} CTOD \approx \frac{G}{\sigma_{yld}}$$

Second-order (Dugdale) CTOD

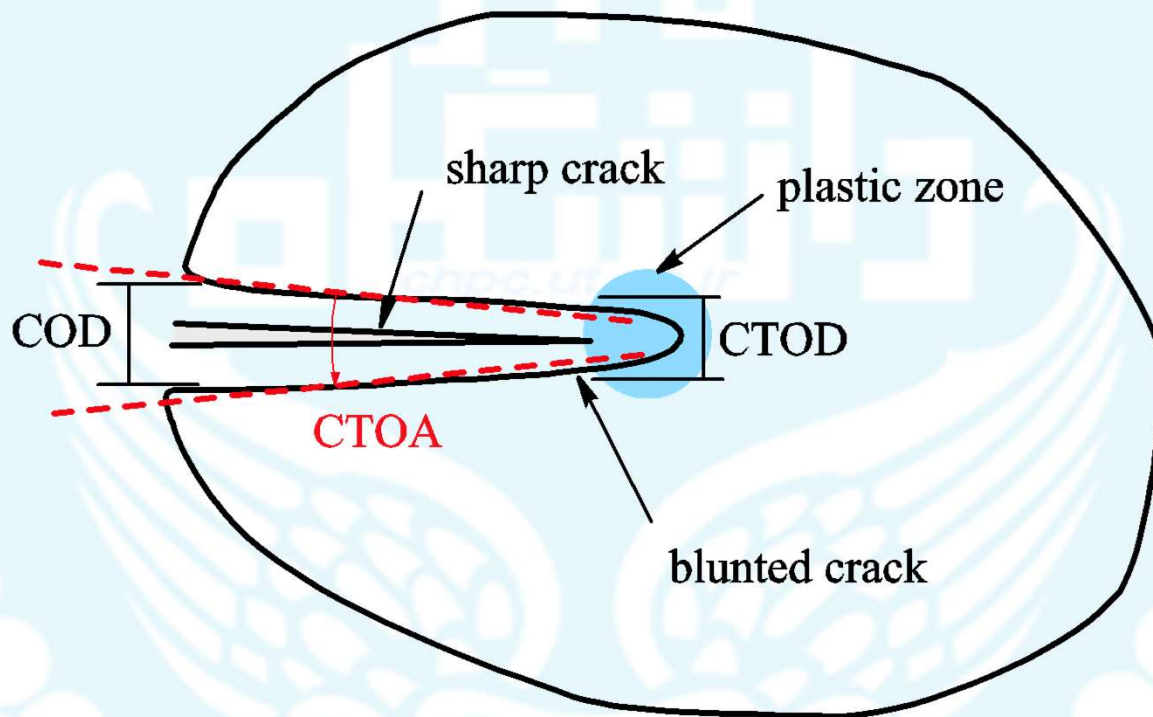
The second-order CTOD formulation is based on the application of the second-order Dugdale model along the crack for determination of COD (Kanninen, 1984):

$$u_y(x) = \frac{2}{\pi} \frac{a\sigma_{yld}}{E} \left\{ \ln \left| \frac{e + \sqrt{c^2 - x^2}}{e - \sqrt{c^2 - x^2}} \right| + \frac{x}{a} \ln \left| \frac{xe + a\sqrt{c^2 - x^2}}{xe - a\sqrt{c^2 - x^2}} \right| \right\}$$
$$e^2 = c^2 - a^2, \quad 0 \leq x \leq c$$

and for $x=a$:

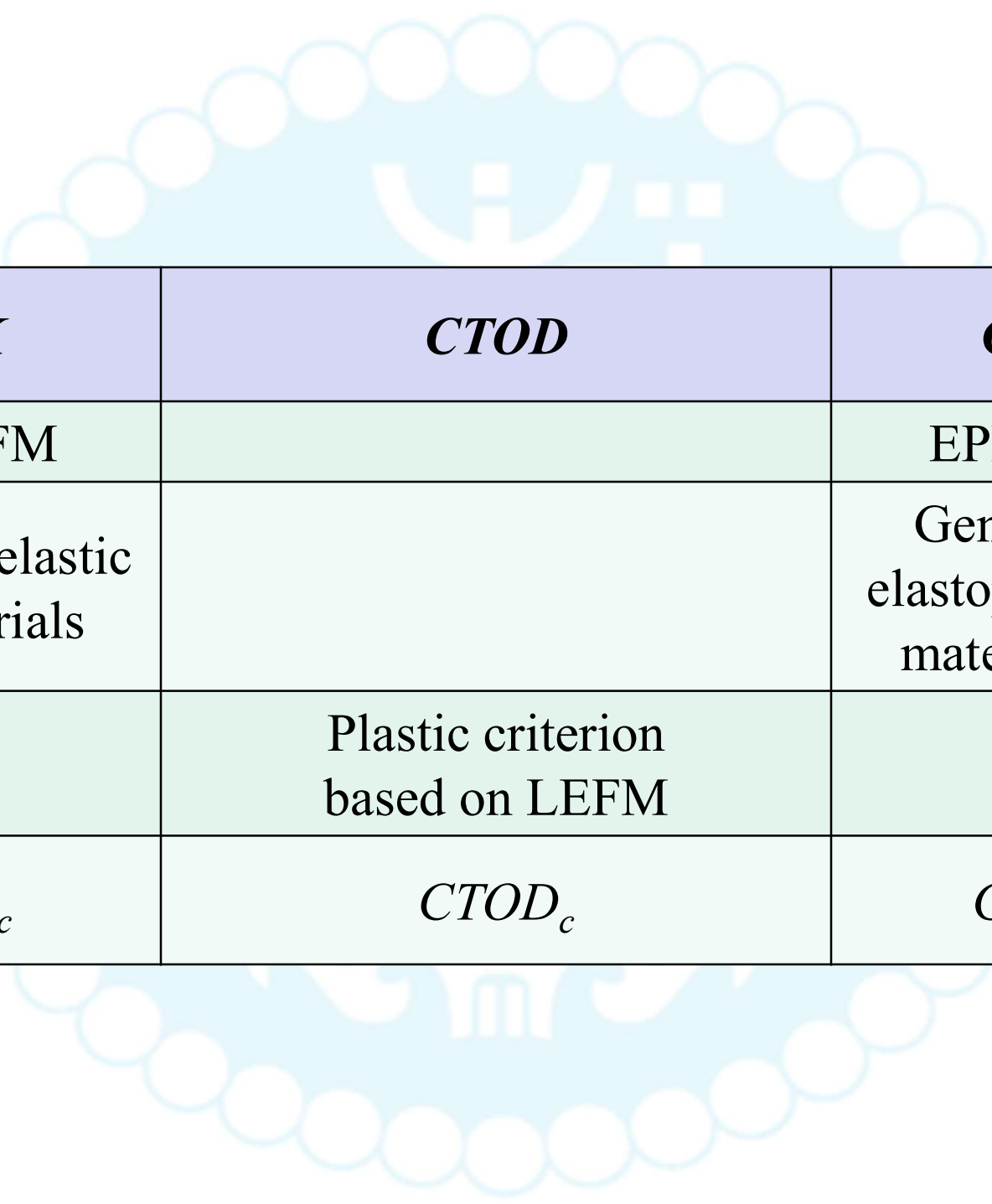
$$\begin{aligned} \text{CTOD} = 2u_y(x=a) &= \frac{8}{\pi} \frac{a\sigma_{yld}}{E} \log \left[\sec \frac{\pi}{2} \frac{\sigma}{\sigma_{yld}} \right] \\ &= \frac{K^2}{E\sigma_{yld}} \left[1 + \frac{\pi^2}{24} \frac{\sigma^2}{\sigma_{yld}^2} + \dots \right] \end{aligned}$$

Sometimes, crack tip opening angle (CTOA) is also determined and compared to its critical value, $CTOA_c$, recommended by the code of practice or experimental tests.



CTOD, COD and CTOA based criteria:

- They are used in engineering fracture mechanics problems.
- They are rarely adopted for academic research purposes.
- They may clearly fail for general crack propagation problems.
- They have to be used only in problems with small-scale plastic zones around the crack tip.



K	$CTOD$	G
LEFM		EPFM
Linear elastic materials		General elastoplastic materials
	Plastic criterion based on LEFM	
K_c	$CTOD_c$	G_c

